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A Negative Result on Multivariate Convex Approximation by Positive Linear Operators

Xuguang Lu

Department of Applied Mathematics, Tsinghua University, Beijing 100084, People's Republic of China

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We consider the problem of multivariate convex approximation by positive linear operators. Let E be a k-dimensional compact convex set in $\mathbf{R}^k$ with $k \geqslant 2$, $\Omega \subset \mathbf{R}^k$ an open set containing E, and let $L: C(E) \to C^1(\Omega)$ be a positive linear operator. Our main result of this paper shows that if L preserves convexity and satisfies $Ll = l$ on E for all $l \in \mathbf{P}_1$ (the space of affine functions), then L is trivial (i.e., $Lf \in \mathbf{P}_1$ on E for all $f \in C(E)$) and E is a simplex. © 1996 Academic Press, Inc.

1. INTRODUCTION AND MAIN RESULT

For multivariate polynomial convex approximation, some fundamental results have been obtained by A. S. Švedov [1] with Jackson type estimates (see also [2]). More general results on shape preserving approximation by multivariate polynomials have been given in [3]. But for multivariate convex approximation by positive linear operators, even the following problem has not been solved:

Open Problem. *Do there exist a k-dimensional compact convex set $E \subset \mathbf{R}^k$ with $k \geqslant 2$, and a sequence $L_n: C(E) \to C^1(\Omega_n)$ of positive linear operators, where $\Omega_n \subset \mathbf{R}^k$ are open sets containing E, such that each L_n preserves convexity and $\lim_{n\to\infty} \|L_n f - f\|_E = 0$ for all $f \in C(E)$?*

Here $C(E)$ denotes the Banach space of real continuous functions defined on E with the maximum norm $\|\cdot\|_E$: $\|f\|_E = \max\{|f(x)| \mid x \in E\}$, $\mathbf{P}_n$ denotes the space of k-variable real polynomials with total degree $\leqslant n$. For a linear operator $L: C(E) \to C(E)$, if $f \geqslant 0$ on $E \Rightarrow Lf \geqslant 0$ on E, then L is called positive; if f is convex on $E \Rightarrow Lf$ is also convex on E, then we say that L preserves convexity; if $(Lf)|_E \in \mathbf{P}_1$ for all $f \in C(E)$, then we say that L is trivial. Trivial operators cannot be used to approximate continuous functions. In [2], the author gave a partial negative answer to the problem for polynomial operator cases: If for some $n \geqslant 2$, $L_n: C(E) \to \mathbf{P}_n$

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is a non-trivial positive linear operator and is invariant on affine functions, i.e., $L_n l = l$ for all $l \in \mathbf{P}_1$, then L_n can not preserve convexity. Note that when $k=1$, there is no such problem; the Bernstein operators $B_n\colon C([0,1]) \to \mathbf{P}_n$ preserve convexity, monotonicity, etc., and hold $\lim_{n\to\infty} \|B_n f - f\|_{[0,1]} = 0$ for all $f \in C([0,1])$ as well known. Then we see that there exist some essential differences between the cases of one variable and multivariable in (polynomial) positive linear operator shape preserving approximation. These differences were first observed by Chang and Davis [4] for Bernstein polynomial operator cases; they exhibited (by computing) a simple bivariate convex (piecewise linear) function whose second degree Bernstein polynomial is not convex. As commented in [5, § 4], this observation triggered numerous studies of convexity preserving properties of the Bernstein–Bézier representation of multivariate polynomials with many positive and constructive results. This investigation process shows again that negative results are also often promote the development of the subject in positive direction. In [2], two of key steps for proving the negative result mentioned above were based on Markov inequalities and analyticity of polynomials. Through further investigation we find that the restriction that L maps $C(E)$ into a polynomial space is not necessary. The important factors that influence the convexity preserving property are (or at least include) the following three aspects:

(1) the positivity of a positive linear operator L,

(2) the behavior of L acting on affine function space $\mathbf{P}_1$ and geometric property of E,

(3) the regularity of L, i.e., L maps $C(E)$ into a smooth function space, for instance $C^1(\Omega)$, where Ω is an open set containing E.

On the aspect (1), the author in [6] (see also [3]) constructed a kind of Bernstein–Durrmeyer type polynomial operators $\mathscr{M}_n$, $\mathscr{M}_{n-s,s}\colon C(T^*) \to \mathbf{P}_n$ defined by $\mathscr{M}_n = \mathscr{M}_{n,0}$,

$$\mathscr{M}_{n-s,s} f(x) = \frac{(kn+k+s)!}{(kn+s)!} \sum_{|v| \leqslant n-s} B_{n-s,v}(x) \int_T B_{kn+s,\,ne^*-v}(t)\, f(e^*-kt)\, dt,$$

$s = 0, 1, ..., n$, where $B_{n,v}$ $(\equiv P_{n,v}) \in \mathbf{P}_n$ are Bernstein base functions, $e^* = (1, 1, ..., 1)$,

$$T = \left\{ x = (x_1, x_2, ..., x_k) \mid x_i \geqslant 0,\ i = 1, 2, ..., k;\ \sum_{i=1}^{k} x_i \leqslant 1 \right\},$$

$$T^* = \left\{ x = (x_1, x_2, ..., x_k) \mid 1-k \leqslant x_i \leqslant 1,\ i = 1, 2, ..., k;\ \sum_{i=1}^{k} x_i \geqslant 0 \right\}$$
$$= \{ e^* - kt \mid t \in T \} \supset T, \qquad k \geqslant 2.$$

The operators $\mathscr{M}_n$, $\mathscr{M}_{n-s,s}$ satisfy $\mathscr{M}_n 1 = 1$, $\mathscr{M}_{n-s,s} 1 = 1$ and possess the following important property: Let $P(D) = \sum_{|\alpha|=s} C_\alpha D^\alpha$ be a homogeneous partial differential operator. Then for $n > s$,

$$P(D)(\mathscr{M}_n f) = a_{n,s} \mathscr{M}_{n-s,s}(P(D) f), \qquad \forall f \in C^s(T^*), \tag{1.1}$$

where

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_k^{\alpha_k}, \qquad \alpha = (\alpha_1, \alpha_2, ..., \alpha_k),$$

$$D_i = \partial/\partial x_i, \qquad a_{n,s} = k^s n!\,(kn+k)!/(n-s)!\,(kn+k+s)!.$$

Using the property (1.1) we have proved in [6] that if $f \in C(T^*)$ is convex on T^*, then $\mathscr{M}_n f$ is convex on T, and $\lim_{n\to\infty} \|\mathscr{M}_n f - f\|_T = 0$ for all $f \in C(T^*)$. But these operators are not positive on $C(T^*)$. That is , for $f \geqslant 0$ on T^*, we only have $\mathscr{M}_n f \geqslant 0$, $\mathscr{M}_{n-s,s} f \geqslant 0$ on T. The operator $\mathscr{M}_n$ is a modification of the Bernstein–Durrmeyer operator M_n: $C(T) \to \mathbf{P}_n$, defined by (see [7], [8])

$$M_n f(x) = \frac{(n+k)!}{n!} \sum_{|v| \leqslant n} B_{n,v}(x) \int_T B_{n,v}(t) f(t)\,dt.$$

The operators M_n are positive on $C(T)$ and hold $\lim_{n\to\infty} \|M_n f - f\|_T = 0$ for all $f \in C(T)$, but they cannot preserve convexity. In fact, if (for some n) M_n preserves convexity, then it must be a trivial operator , or equivalently, $n = 1$. To see this, we choose a special convex function $g(x) = |x_1 - x_2|$ ($x = (x_1, x_2, ..., x_k)$) and suppose that M_n preserves convexity. The following equalities can be easily checked by computing and properties of M_n (see [7], [8]).

$$M_n g(0) = \frac{1}{n+k+1}, \qquad D_1 M_n g(0) = D_2 M_n g(0) = \frac{1}{2} \cdot \frac{n}{n+k+1},$$

$$D_j M_n g(0) = 0 \qquad \text{for} \quad j > 2,$$

$$\int_T M_n g(x)\,dx = \int_T g(x)\,dx = \frac{1}{(k+1)!} = \int_T [M_n g(0) + \langle \nabla M_n g(0), x \rangle]\,dx. \tag{1.2}$$

Here and later we denote as usual

$$\nabla f(x) := (D_1 f(x), D_2 f(x), ..., D_k f(x)),$$

$$\langle x, y \rangle := \sum_{i=1}^{k} x_i y_i, \qquad \|x\| := \sqrt{\langle x, x \rangle}.$$

Since the polynomial $M_n g$ is convex on T, we have

$$M_n g(x) \geqslant M_n g(0) + \langle \nabla M_n g(0), x \rangle, \qquad \forall x \in T,$$

and then by (1.2),

$$M_n g(x) = M_n g(0) + \langle \nabla M_n g(0), x \rangle, \qquad \forall x \in T.$$

Especially for $x = e_1 = (1, 0, ..., 0)$,

$$M_n g(e_1) = M_n g(0) + D_1 M_n g(0) = \frac{1}{n+k+1}\left(1 + \frac{1}{2}n\right).$$

On the other hand, by the choice of g and $M_n l_i(x) = (1 + nx_i)/(n+k+1)$ we have

$$M_n g(e_1) > M_n(l_1 - l_2)(e_1) = \frac{n}{n+k+1},$$

where $l_i(x) = x_i$ for $x = (x_1, x_2, ..., x_k)$. Thus $1 + \frac{1}{2}n > n$ and so $n = 1$.

On the aspects (2), (3), we have the following theorem which is the main result of this paper and gives a further partial negative answer to the open problem.

THEOREM. *Let $E \subset \mathbf{R}^k$ be a k-dimensional compact convex set with $k \geqslant 2$, $\Omega \subset \mathbf{R}^k$ an open set containing E, and let $L\colon C(E) \to C^1(\Omega)$ be a positive linear operator satisfying*

(i) $L1 = 1$,

(ii) $\forall l \in \mathbf{P}_1$, $Ll \geqslant 0$ *on* $E \Rightarrow l \geqslant 0$ *on* E.

Then

(a) *E is a convex polyhedron.*

(b) *L can not preserve convexity unless L is trivial and E is a simplex.*

Remarks. The condition (ii) is equivalent to geometric condition $\operatorname{conv} \sigma(E) = E$ under the condition (i) (see the proof of the Theorem in § 3), where $\sigma(x) = (Ll_1(x), Ll_2(x), ..., Ll_k(x))$, $l_i(x) = x_i$. If E is already a k-dimensional compact convex polyhedron, and $a_0, a_1, ..., a_m$ are all its extreme points, then the following condition

$$Ll(a_i) = l(a_i),\ i = 0, 1, ..., m, \qquad \forall l \in \{l_1, l_2, ..., l_k\}, \tag{*}$$

together with (i), implies condition (ii).

Obviously, if L is invariant on $\mathbf{P}_1$, the condition (i), (ii) are both satisfied. In applications, this Theorem is more useful than those obtained in [2]. For instance , if E is a k-dimensional compact convex polyhedron ($k \geqslant 2$) and L is a non-trivial positive linear operator from $C(E)$ to a C^1-spline function space or a polynomial space and satisfies conditions (i) and (*) (therefore condition (ii)), then L can not preserve convexity. As a consequence, all the Bernstein–Bézier polynomial operators (see [5]) of degree $\geqslant 2$ over k-dimensional simplices or rectangles with $k \geqslant 2$ can not preserve convexity. We can also get even more general negative conclusion on such operators when using condition (ii) (or sufficiently condition(*)). Let L_n: $C(T) \to \mathbf{P}_n$ be Bernstein type operator of degree $n \geqslant 2$ having the form

$$L_n f(x) = \sum_{|v| \leqslant n} B_{n,v}(x)\, \Lambda_{n,v} f, \tag{1.3}$$

where $\Lambda_{n,v}$ are positive linear functionals defined on $C(T)$ satisfying $\Lambda_{n,v} 1 = 1$ $\forall v$ and $\Lambda_{n, na_i} l_j = \delta_{i,j}$ (Kronecker delta), $i = 0, 1, ..., k$; $j = 1, 2, ..., k$, where $a_0 = (0, ..., 0)$, $a_1 = (1, 0, ..., 0)$, ..., $a_k = (0, ..., 0, 1)$, $\operatorname{conv}\{a_0, a_1, ..., a_k\} = T$. By Bernstein polynomial property we have $L_n 1 = 1$, $L_n l(a_i) = l(a_i)$, $i = 0, 1, ..., k$ for all $l \in \{l_1, l_2, ..., l_k\}$. Therefore L_n satisfies condition (ii). Note that L_n needs not to be invariant on $\mathbf{P}_1$. For instance, if there exist some $l \in \mathbf{P}_1$ and $v/n \notin \{a_0, a_1, ..., a_k\}$ such that $\Lambda_{n,v} l \neq l(v/n)$, then $L_n l \neq l$ by Bernstein polynomial property. Many concrete examples of such operators can be easily constructed without loss approximation property (see example below). But, of course, they can not preserve convexity.

For C^1-spline operator cases, however, we could not find any known such operator that satisfies conditions (i), (*) (or conditions (i), (ii)) and so does not preserve convexity and this was not known before. To the author's knowledge, there was perhaps no such C^1-spline positive linear operator. If it is so, then the Theorem of the present paper may be only or at least useful for future theoretical applications.

Finally we remark that since some known positive linear operators (e.g., Bernstein–Durremeyer operator M_n discussed above) neither satisfy condition (ii) nor preserve convexity, whether the condition (ii) can be left out from the Theorem when E is already supposed to be a k-dimensional simplex remains open to investigate.

EXAMPLE. We present here a simple method to obtain a class of positive linear operators L_n: $C(T) \to \mathbf{P}_n$ that satisfy conditions (i) and (*) (therefore condition (ii)) and are not invariant on $\mathbf{P}_1$, and possess approximation property. Let $\tilde{\Lambda}_{n,v}$ be positive functionals on $C(T)$ defined as

$$\tilde{\Lambda}_{n,v} \in \{\Lambda^{(1)}_{n,v}, \Lambda^{(2)}_{n,v}, ...\}, \qquad \forall\, |v| \leqslant n, \quad n \geqslant 2,$$

where

$$\Lambda_{n,v}^{(1)} f = f\left(\frac{v+e^*}{n+k}\right), \qquad e^* = (1, 1, ..., 1) \in \mathbf{R}^k, \quad (k \geqslant 2)$$

$$\Lambda_{n,v}^{(2)} f = \frac{(n+k)!}{n!} \int_T B_{n,v}(t) f(t)\, dt,$$

$$\cdots$$

so that $\tilde{\Lambda}_{n,v}$ satisfy

(a) $\tilde{\Lambda}_{n,v} 1 = 1, \ \forall\ |v| \leqslant n$,

(b) $\exists l \in \mathbf{P}_1, \ \exists v\colon |v| \leqslant n, \ v/n \notin \{a_0, a_1, ..., a_k\}$ such that $\tilde{\Lambda}_{n,v} l \neq l(v/n)$,

(c) $\lim_{n \to \infty} \|\tilde{L}_n f - f\|_T = 0$ for all $f \in C(T)$, where

$$\tilde{L}_n f(x) = \sum_{|v| \leqslant n} B_{n,v}(x)\, \tilde{\Lambda}_{n,v} f.$$

Define

$$\Lambda_{n,v} f = \begin{cases} \tilde{\Lambda}_{n,v} f, & |v| \leqslant n, \quad \dfrac{v}{n} \notin \{a_0, a_1, ..., a_k\} \\ f(a_i), & v = n a_i, \quad i = 0, 1, ..., k, \end{cases}$$

and define L_n by (1.3). Then L_n are positive linear operators form $C(T)$ into $\mathbf{P}_n$ satisfying $L_n 1 = 1$, $L_n f(a_i) = f(a_i)\ \forall f \in C(T)$, $\forall i = 0, 1, ..., k$ and $L_n l \neq l$ for some $l \in \mathbf{P}_1$ by Bernstein polynomial properties. These show that L_n are not invariant on $\mathbf{P}_1$ and satisfy conditions (i) and (*). Also, we have $\|L_n f - f\|_T \leqslant 2\, \|\tilde{L}_n f - f\|_T \to 0 \ \ (n \to \infty)$ for all $f \in C(T)$. In fact, by Bernstein polynomial properties we have $\tilde{L}_n f(a_i) = \tilde{\Lambda}_{n, na_i} f$, $i = 0, 1, ..., k$, which imply

$$\begin{aligned} \|L_n f - \tilde{L}_n f\|_T &= \left\| \sum_{|v| \leqslant n} B_{n,v}(\cdot)[\Lambda_{n,v} f - \tilde{\Lambda}_{n,v} f] \right\|_T \\ &\leqslant \max_{0 \leqslant i \leqslant k} |f(a_i) - \tilde{\Lambda}_{n, na_i} f| \leqslant \|f - \tilde{L}_n f\|_T. \quad \blacksquare \end{aligned}$$

2. SOME LEMMAS

This section collects some lemmas for the proof of the Theorem.

LEMMA 1. *Let $E \subset \mathbf{R}^k$ be a compact convex set, $\Lambda\colon C(E) \to \mathbf{R}$ a positive linear functional with $\Lambda 1 = 1$. Define $\sigma = (\Lambda l_1, \Lambda l_2, ..., \Lambda l_k)$, where $l_i(x) = x_i (x = (x_1, x_2, ..., x_k))$. Then $\sigma \in E$ and $\Lambda f \geqslant f(\sigma)$ for all convex function $f \in C(E)$.*

Proof. Let $f \in C(E)$ be convex and let $\mathscr{F} = \{(x, u) \in \mathbf{R}^k \times \mathbf{R} \mid x \in E, u \geqslant f(x)\}$. Then $\mathscr{F}$ is a closed convex set. We need to prove $(\sigma, \Lambda f) \in \mathscr{F}$. Suppose that $(\sigma, \Lambda f) \notin \mathscr{F}$, then by separation theorem, there exists a point $(x_0, u_0) \in \mathscr{F}$, such that

$$\langle (\sigma, \Lambda f) - (x_0, u_0), (x, u) - (x_0, u_0) \rangle \leqslant 0 \qquad \text{for all} \quad (x, u) \in \mathscr{F}.$$

Define $g(x) = \langle (\sigma, \Lambda f) - (x_0, u_0), (x, f(x)) - (x_0, u_0) \rangle$. Then $g(x) \leqslant 0$ for all $x \in E$ since $(x, f(x)) \in \mathscr{F}$. Thus we obtain a contradiction: $0 \geqslant \Lambda g = \|(\sigma, \Lambda f) - (x_0, u_0)\|^2 > 0$, and the Lemma is proved. ∎

LEMMA 2. *Let $E \subset \mathbf{R}^k$ be a k-dimensional compact convex set, $\Omega \subset \mathbf{R}^k$ an open set containing E. Suppose that $L: C(E) \to C^1(\Omega)$ is a linear operator which is bounded when considered as operator from $C(E)$ into $C(E)$. Then the operators $D_i L: C(E) \to C(E)$ are also bounded for all $i = 1, 2, ..., k$.*

Proof. Equivalently , we prove that there exists a positive constant M depending only on E and L such that

$$\max_{x \in E} \|\nabla Lf(x)\| \leqslant M \|f\|_E, \qquad \forall f \in C(E). \tag{2.1}$$

Let $W = \{(x, y) \mid x, y \in E, x \neq y\}$. For any $\omega = (x, y) \in W$, define

$$T_\omega f = \frac{1}{\|y - x\|} [Lf(y) - Lf(x)], \qquad f \in C(E).$$

Then T_ω is a bounded linear functional on $C(E)$,

$$\|T_\omega\| := \sup\{|T_\omega f| / \|f\|_E \mid f \in C(E), \|f\|_E \neq 0\} \leqslant \frac{2}{\|y - x\|} \|L\|,$$

where

$$\|L\| = \sup\{\|Lf\|_E / \|f\|_E \mid f \in C(E), \|f\|_E \neq 0\} < \infty.$$

Since E is convex and $Lf \in C^1(\Omega)$ for all $f \in C(E)$, it follows that $\forall \omega \in W$, $|T_\omega f| \leqslant \max_{\xi \in E} \|\nabla Lf(\xi)\|$ and so $\sup\{|T_\omega f| \mid \omega \in W\} < \infty$. By the Banach–Steinhaus resonance theorem, the set $\{\|T_\omega\| \mid \omega \in W\}$ is bounded, i.e., $\exists M > 0$ such that $\|T_\omega\| \leqslant M$ for all $\omega \in W$. These yield $\|\nabla Lf(x)\| \leqslant M \|f\|_E$ for all $x \in \text{int } E$ and all $f \in C(E)$, and (2.1) holds since $Lf \in C^1(\Omega)$ and $\text{cl}(\text{int } E) = E$ (see [9, § 6]) $\subset \Omega$. ∎

LEMMA 3 (Linear Inequalities [9, p. 198, Theorem 22.1]). *Let $b_i \in \mathbf{R}^k$ and $\alpha_i \in \mathbf{R}$ for $i = 1, 2, ..., m$. Then one and only one of the following alternatives holds:*

(a) *There exists a vector* $x \in \mathbf{R}^k$ *such that* $\langle b_i, x \rangle \leqslant \alpha_i$, $i = 1, 2, ..., m$.

(b) *There exist non-negative real numbers* $\lambda_1, \lambda_2, ..., \lambda_m$ *such that*

$$\sum_{i=1}^{m} \lambda_i b_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{m} \lambda_i \alpha_i < 0. \quad \blacksquare$$

The last lemma below looks very natural in geometry for $k = 2$, but we could not find any known reference for general case to such lemma.

LEMMA 4. *Let* $E \subset \mathbf{R}^k$ *be a* k*-dimensional compact convex polyhedron with* $k \geqslant 2$, *and* $D(E)$ *denote the set of extreme points of* E. *Then for any* $a_0 \in D(E)$, there exists a subset $\{a_0, a_1, ..., a_k\} \subset D(E)$ *and a convex function* $\varphi \in C(\mathbf{R}^k)$ *such that*

(a) $a_0, a_1, ..., a_k$ *are affinely independent*,

(b) $\varphi(x) \leqslant 0$ *for all* $x \in E$; $\varphi(x) = 0$, $x \in E$ *if and only if* $x \in$ *the edge set* $\Gamma(a_0)$, *where*

$$\Gamma(a_0) = \bigcup_{i=0}^{k} \{a_0 + \tau(a_i - a_0) \mid \tau \in [0, 1]\}.$$

Proof. Since E is a compact convex polyhedron, the set $D(E)$ is finite. Let $a_0 \in D(E)$ and write $D(E) = \{a_0, a_1, ..., a_m\}$. By definition of extreme point, it is easily seen that the following properties hold:

$$\textit{If } \lambda_i \geqslant 0,\ i = 1, 2, ..., m,\ \sum_{i=1}^{m} \lambda_i (a_i - a_0) = 0,\ \textit{then } \lambda_i = 0,\ i = 1, 2, ..., m. \tag{2.2}$$

$$\textit{If } 0 < i < j,\ \textit{then } a_i - a_0 \textit{ and } a_j - a_0 \textit{ are linearly independent.} \tag{2.3}$$

Define $b_i = a_i - a_0$, $K = \{\sum_{i=1}^{m} \lambda_i b_i \mid \lambda_i \geqslant 0, i = 1, 2, ..., m\}$, $K(I) = \{\sum_{i \in I} \lambda_i b_i \mid \lambda_i \geqslant 0,\ \forall i \in I\}$, where $I \subset \{1, 2, ..., m\}$. Let $r = min\{|I| \mid I \subset \{1, 2, ..., m\}, K(I) = K\}$. Choose $I_0 \subset \{1, 2, ..., m\}$ such that $|I_0| = r$ and $K(I_0) = K$. Without loss of generality, we can suppose that $I_0 = \{1, 2, ..., r\}$. Since $\dim E = k$, it is easy to see that $r \geqslant k (\geqslant 2)$ and we can suppose that b_1, $b_2, ..., b_k$ are linearly independent, which imply that $a_0, a_1, ..., a_k$ are affinely independent. By (2.2) and the minimality of $|I_0|$, the following linear inequality system

$$\begin{cases} \lambda_0(-b_1) + \lambda_1 b_1 + \cdots + \lambda_r b_r = 0, \\ -\lambda_2 - \lambda_3 - \cdots - \lambda_r < 0, \end{cases}$$

has no non-negative solution. Thus by Lemma 3 there exists a point $p_1 \in \mathbf{R}^k$ such that $\langle -b_1, p_1 \rangle \leqslant 0$, $\langle b_1, p_1 \rangle \leqslant 0$, $\langle b_i, p_1 \rangle \leqslant -1$, $i = 2, 3, ..., r$. If $r < m$ and $r < j \leqslant m$, then $b_j = \sum_{i=1}^{r} \lambda_{i,j} b_i$ with $\lambda_{i,j} \geqslant 0$, $i = 1, 2, ..., r$ and

$\sum_{i=2}^{r} \lambda_{i,j} > 0$ by (2.3). These imply $\langle b_j, p_1 \rangle = \sum_{i=2}^{r} \lambda_{i,j} \langle b_i, p_1 \rangle < 0$. We have proved that $\langle b_1, p_1 \rangle = 0$, $\langle b_j, p_1 \rangle < 0$, $j = 2, 3, ..., m$. Using the same argument for each $i \in \{1, 2, ..., r\}$ we obtain that there exist points $p_i \in \mathbf{R}^k$ such that

$$\langle b_i, p_i \rangle = 0, \qquad \langle b_j, p_i \rangle < 0, \qquad j = 1, 2, ..., m, \; j \neq i. \tag{2.4}$$

Now let $\varphi(x) = \max\{\langle x - a_0, p_1 \rangle, \langle x - a_0, p_2 \rangle, ..., \langle x - a_0, p_k \rangle\}$. Then it is easy to check by $E = \mathrm{conv}\{a_0, a_1, ..., a_m\}$ and (2.4) that φ is the desired convex function. ▮

3. PROOF OF THE THEOREM

(a) For any $x \in E$, $L(\cdot)(x)$ is a positive linear functional on $C(E)$. By Riesz's representation theorem,

$$Lf(x) = \int_E f(t)\, \mu_x(dt), \qquad \forall f \in C(E), \quad x \in E. \tag{3.1}$$

where μ_x is a positive Borel measure. Define $\sigma: \Omega \to \mathbf{R}^k$,

$$\sigma(x) = (Ll_1(x), Ll_2(x), ..., Ll_k(x))$$

where $l_i(x) = x_i$ for $x = (x_1, x_2, ..., x_k)$. Then σ is continuous in Ω and $\sigma(E) \subset E$ by Lemma 1. We first prove that

$$\mathrm{conv}\, \sigma(E) = E. \tag{3.2}$$

Suppose, to the contrary, that there exists a point $x^* \in E \backslash \mathrm{conv}\, \sigma(E)$. By separation theorem (note that the convex hull $\mathrm{conv}\, \sigma(E)$ is also compact) $\exists a \in \mathrm{conv}\, \sigma(E)$ such that $\langle x - a, x^* - a \rangle \leqslant 0$ for all $x \in \sigma(E)$. Take $l(x) = -\langle x - a, x^* - a \rangle$. Then $Ll = l \circ \sigma \geqslant 0$ on E and so by condition (ii), $l \geqslant 0$ on E. This contradicts to $l(x^*) < 0$. (3.2) holds. Now let $D(E)$ and $D_0(E)$ denote the sets of the extreme points and the exposed points of E respectively. Then by (3.2) we have

$$E = \mathrm{conv}\, D(E), \qquad D_0(E) \subset D(E) \subset \sigma(E). \tag{3.3}$$

For any exposed point $z \in D_0(E)$, there exists a point $p \in \mathbf{R}^k$ such that the affine function $l(x) := \langle x - z, p \rangle < 0$ for all $x \in E$ with $x \neq z$. (see [9, pp. 162–163]). Let $\hat{z} \in \sigma^{-1}(\{z\}) = \{x \in E \mid \sigma(x) = z\}$ ((3.3) insures that $\sigma^{-1}(\{z\})$ is non-empty). We have, by (3.1),

$$\int_E l(t)\, \mu_{\hat{z}}(dt) = l(\sigma(\hat{z})) = l(z) = 0$$

and so the measure $\mu_{\hat{z}}$ concentrates on the point z, i.e., $\mu_{\hat{z}}$ is the unit mass concentrated at z. Thus

$$Lf(\hat{z}) = f(z) \qquad \text{for all} \quad f \in C(E),\, z \in D_0(E), \text{ and } \hat{z} \in \sigma^{-1}(\{z\}). \quad (3.4)$$

To prove E is a convex polyhedron, we need only to prove that the set $D(E)$ is finite. Since, by Straszewicz's theorem (see [9, p. 167]), the set $D_0(E)$ is dense in $D(E)$, we need only to prove that $D_0(E)$ is finite. Let $x, y \in D_0(E)$, $x \neq y$. By Tietze's extension theorem, there exists an $f \in C(E)$ such that $f(x) = 1$, $f(y) = -1$ and $\|f\|_E = 1$. For any $\hat{x} \in \sigma^{-1}(\{x\})$, $\hat{y} \in \sigma^{-1}(\{y\})$ we have, by (3.4) and Lemma 2,

$$\begin{aligned} 2 &= f(x) - f(y) = Lf(\hat{x}) - Lf(\hat{y}) = \langle \nabla Lf(\xi), \hat{x} - \hat{y} \rangle \\ &\leqslant M \,\|\hat{x} - \hat{y}\|, \end{aligned}$$

which imply $\operatorname{dist}(\sigma^{-1}(\{x\}), \sigma^{-1}(\{y\})) \geqslant 2M^{-1}$, where the constant M depends only on E and L. Hence the set $D_0(E)$ is finite and so $D(E) = D_0(E)$.

(b) Suppose that the operator L preserves convexity. Then L preserves linearity, i.e., $l \in \mathbf{P}_1 \Rightarrow (Ll)|_E \in \mathbf{P}_1$. This leads to

$$\forall x \in E, \quad \sigma(x) = xA + b, \qquad \sigma(E) = E, \qquad \text{and} \qquad \det A \neq 0 \quad (3.5)$$

by (3.2), where A is a $k \times k$-dimensional constant matrix, b is a constant vector. (Since $\dim E = k$, we have $|\det A|\, mes(E) = mes(\sigma(E)) = mes(E) > 0$, so $\det A \neq 0$.) Define $L_\sigma \colon C(E) \to C^1(\Omega)$,

$$L_\sigma f(x) = L(f \circ \sigma^{-1})(x), \qquad (\sigma^{-1}(x) := (x - b)\, A^{-1} \textit{ for all } x \in E)$$

Clearly, L_σ is a positive linear operator preserving convexity and satisfies $L_\sigma l = l$ on E for all $l \in \mathbf{P}_1$. Moreover (3.5) implies $\sigma(D(E)) = D(E)$, so by (3.4) and $D(E) = D_0(E)$ we obtain

$$L_\sigma f(z) = f(z) \qquad \text{for all} \quad f \in C(E), \quad z \in D(E). \qquad (3.6)$$

Since E is a k-dimensional compact convex polyhedron, it follows by Lemma 4 that for any $a_0 \in D(E)$, there exist points $a_1, a_2, ..., a_k \in D(E)$ and a convex function $\varphi \in C(\mathbf{R}^k)$ such that Lemma 4 (a) and (b) hold. Using Lemma 1 and Lemma 4(b) we obtain $0 \geqslant L_\sigma \varphi(x) \geqslant \varphi(x) = 0$ for all $x \in \Gamma(a_0)$, i.e.,

$$L_\sigma \varphi(a_0 + \tau(a_i - a_0)) = 0, \qquad \forall \tau \in [0, 1],\; i = 1, 2, ..., k.$$

These imply by Lemma 4(a) that $D_i L_\sigma \varphi(x)|_{x=a_0}=0$, $i=1, 2, ..., k$. Since $L_\sigma \varphi$ is convex on E and $L_\sigma \varphi \in C^1(\Omega)$, it follows that

$$0 \geqslant L_\sigma \varphi(x) \geqslant L_\sigma \varphi(a_0) + \langle \nabla L_\sigma \varphi(a_0), x - a_0 \rangle = 0$$

i.e.,

$$L_\sigma \varphi(x) = \int_E \varphi(\sigma^{-1}(t))\, \mu_x(dt) = 0, \qquad \forall x \in E. \tag{3.7}$$

Especially, by (3.6), for any $z \in D(E)$, $\varphi(z) = L_\sigma \varphi(z) = 0$, and so $z \in \{a_0, a_1, ..., a_k\}$ since z is an extreme point. Thus $D(E) = \{a_0, a_1, ..., a_k\}$ and E is a simplex. Furthermore, (3.5), (3.7), and Lemma 4(b) imply $\mu_x(E \backslash \sigma(\Gamma(a_0))) = 0$ for all $x \in E$. Since $a_0 \in D(E)$ is arbitrary, substituting a_s for a_0 we also obtain $\mu_x(E \backslash \sigma(\Gamma(a_s))) = 0$ for all $x \in E$, $s = 1, 2, ..., k$, where $\Gamma(a_s)$ are edge sets of E:

$$\Gamma(a_s) = \bigcup_{i=0}^{k} \{a_s + \tau(a_i - a_s) \mid \tau \in [0, 1]\}, \qquad s = 0, 1, ..., k.$$

Because $a_0, a_1, ..., a_k$ are affinely independent and $k+1 \geqslant 3$, the following equality is obvious:

$$\bigcap_{s=0}^{k} \Gamma(a_s) = \{a_0, a_1, ..., a_k\} \qquad (= D(E)). \tag{3.8}$$

Therefore for each $x \in E$, the measure μ_x by (3.5) and (3.8) concentrates on

$$\bigcap_{s=0}^{k} \sigma(\Gamma(a_s)) = \sigma\left(\bigcap_{s=0}^{k} \Gamma(a_s)\right) = \sigma(D(E)) = D(E).$$

Now for any $f \in C(E)$, choose a linear interpolating function l such that $l(a_i) = f(a_i)$, $i = 0, 1, 2, ..., k$. Then by (3.1),

$$Lf(x) = \int_{D(E)} f(t)\, \mu_x(dt) = \int_{D(E)} l(t)\, \mu_x(dt) = Ll(x), \qquad \forall x \in E,$$

and so $(Lf)|_E \in \mathbf{P}_1$ since $(Ll)|_E \in \mathbf{P}_1$ for all $l \in \mathbf{P}_1$. Thus L is trivial and the proof of the Theorem is completed.

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